

# A $5D$ non compact and non Ricci flat Kaluza-Klein Cosmology

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## Abstract

A model universe is proposed in the framework of 5-dimensional noncompact Kaluza-Klein cosmology which is not Ricci flat. The  $4D$  part as the Robertson-Walker metric is coupled to conventional perfect fluid, and its extra-dimensional part is coupled to a dark pressure through a scalar field. It is shown that neither early inflation nor current acceleration of the  $4D$  universe would happen if the non-vacuum states of the scalar field would contribute to  $4D$  cosmology.

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# 1 Introduction

According to the old suggestion of Kaluza and Klein the  $5D$  vacuum Kaluza-Klein equations can be reduced under certain conditions to the  $4D$  vacuum Einstein equations plus the  $4D$  Maxwell equations. Recently, the idea that our four dimensional universe might have emerged from a higher dimensional space-time is receiving much attention [1]. One current interest is to find out in a more general way how the  $5D$  field equations relate to the  $4D$  ones. In this regard, a proposal was made recently by Wesson in that the  $5D$  Einstein equations without sources  $R_{AB} = 0$  ( the Ricci flat assumption ) may be reduced to the  $4D$  ones with sources  $G_{\alpha\beta} = 8\pi GT_{\alpha\beta}$ , provided an appropriate definition is made for the energy-momentum tensor of matter in terms of the extra part of the geometry [2]. Physically, the picture behind this interpretation is that curvature in  $(4+1)$  space induces effective properties of matter in  $(3+1)$  space-time. This idea is known as *space-time-matter* or *modern Kaluza-Klein* theory.

In a parallel way, the brane world scenario [3] assumes that our four-dimensional universe ( the brane ) is embedded in a higher dimensional space-time ( the bulk ). The important ingredient of the brane world scenario, unlike the space-time-matter theory, is that the matter exists apart from geometry and is confined to the brane, and the only communication between the brane and the bulk is through gravitational interaction. The brane world picture relies on a  $Z_2$  symmetry and is inspired from string theory and its extensions [4]. This approach differs from the old Kaluza-Klein idea in that the size of the extra dimensions could be large, more or less similar to the idea in modern Kaluza-Klein theory.

On the other hand, the recent distance measurements of type Ia supernova suggest an accelerating universe [5]. This accelerating expansion is generally believed to be driven by an

energy source which provides positive energy density and negative pressure, such as a positive cosmological constant [6], or a slowly evolving real scalar field called *quintessence* [7]. Since in a variety of inflationary models scalar fields have been used in describing the transition from the quasi-exponential expansion of the early universe to a power law expansion, it is natural to try to understand the present acceleration of the universe by constructing models where the matter responsible for such behavior is also represented by a scalar field. Such models are worked out, for example, in Ref [8]. Bellini *et al*, on the other hand, have published extensively on the evolution of the universe from noncompact *vacuum* Kaluza-Klein theory [11]. They used the essence of STM theory and developed a 5D mechanism to explain, by a single scalar field, the evolution of the universe including inflationary expansion and the present day observed accelerated expansion.

In general, scalar fields are not the only possibility to describe the current acceleration of the universe; there are (of course) alternatives. In particular, one can try to do it by using some perfect fluid but obeying exotic equations of state, the so-called Chaplygin gas [9]. This equation of state has recently raised a certain interest because of its many interesting and, in some sense, intriguingly unique features. For instance, the Chaplygin gas represents a possible unification of dark matter and dark energy, since its cosmological evolution is similar to an initial dust like matter and a cosmological constant for late times [10].

In this paper, motivated by higher dimensional theories, we are interested in constructing a 5D cosmological model which is not Ricci flat, but is extended to be coupled to a higher dimensional energy momentum tensor. This confronts the explicit idea of induced matter in STM theory. Instead, we will show that the higher dimensional sector of this model may

induce a dark pressure, through a scalar field, in four dimensional universe. The implications of this dark pressure on the inflationary phase and current acceleration of the universe will be discussed.

## 2 The Model

We start with the  $5D$  line element

$$dS^2 = g_{AB}dx^A dx^B, \quad (1)$$

in which  $A$  and  $B$  run over both the space-time coordinates  $\alpha, \beta$  and one non compact extra dimension indicated by 4. The space-time part of the metric  $g_{\alpha\beta} = g_{\alpha\beta}(x^\alpha)$  is assumed to define the Robertson-Walker line element

$$ds^2 = dt^2 - R^2(t) \left( \frac{dr^2}{(1 - kr^2)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right), \quad (2)$$

where  $k$  takes the values  $+1, 0, -1$  according to a close, flat or open universe, respectively. We also take the followings

$$g_{4\alpha} = 0, \quad g_{44} = \epsilon \Phi^2(x^\alpha),$$

where  $\epsilon^2 = 1$  and the signature of the higher dimensional part of the metric is left general. This choice has been made because any fully covariant  $5D$  theory has five coordinate degrees of freedom which can lead to considerable algebraic simplification, without loss of generality. Unlike the noncompact vacuum Kaluza-Klein theory, we will assume the fully covariant  $5D$  non-vacuum Einstein equation

$$G_{AB} = 8\pi G T_{AB}, \quad (3)$$

where  $G_{AB}$  and  $T_{AB}$  are the  $5D$  Einstein tensor and energy-momentum tensor, respectively. Note that the  $5D$  gravitational constant has been fixed to be the same value as the  $4D$  one. In the following we use the geometric reduction from  $5D$  to  $4D$  as appeared in [12]. The  $5D$  Ricci tensor is given in terms of the  $5D$  Christoffel symbols by

$$R_{AB} = \partial_C \Gamma_{AB}^C - \partial_B \Gamma_{AC}^C + \Gamma_{AB}^C \Gamma_{CD}^D - \Gamma_{AD}^C \Gamma_{BC}^D. \quad (4)$$

The  $4D$  part of the  $5D$  quantity is obtained by putting  $A \rightarrow \alpha$ ,  $B \rightarrow \beta$  in (4) and expanding the summed terms on the r.h.s by letting  $C \rightarrow \lambda, 4$  etc. Therefore, we have

$$\hat{R}_{\alpha\beta} = \partial_\lambda \Gamma_{\alpha\beta}^\lambda + \partial_4 \Gamma_{\alpha\beta}^4 - \partial_\beta \Gamma_{\alpha\lambda}^\lambda - \partial_\beta \Gamma_{\alpha 4}^4 + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda\mu}^\mu + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda 4}^4 + \Gamma_{\alpha\beta}^4 \Gamma_{4D}^D - \Gamma_{\alpha\lambda}^\mu \Gamma_{\beta\mu}^\lambda - \Gamma_{\alpha\lambda}^4 \Gamma_{\beta 4}^\lambda - \Gamma_{\alpha 4}^D \Gamma_{\beta D}^4, \quad (5)$$

where  $\hat{\phantom{x}}$  denotes the  $4D$  part of the  $5D$  quantities. One finds the  $4D$  Ricci tensor as a part of this equation which may be cast in the following form

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \partial_4 \Gamma_{\alpha\beta}^4 - \partial_\beta \Gamma_{\alpha 4}^4 + \Gamma_{\alpha\beta}^\lambda \Gamma_{\lambda 4}^4 + \Gamma_{\alpha\beta}^4 \Gamma_{4D}^D - \Gamma_{\alpha\lambda}^4 \Gamma_{\beta 4}^\lambda - \Gamma_{\alpha 4}^D \Gamma_{\beta D}^4. \quad (6)$$

Evaluating the Christoffel symbols for the metric  $g_{AB}$  gives

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{\nabla_\alpha \nabla_\beta \Phi}{\Phi}. \quad (7)$$

Putting  $A = 4, B = 4$  and expanding with  $C \rightarrow \lambda, 4$  in Eq.(4) we obtain

$$R_{44} = \partial_\lambda \Gamma_{44}^\lambda - \partial_4 \Gamma_{4\lambda}^\lambda + \Gamma_{44}^\lambda \Gamma_{\lambda\mu}^\mu + \Gamma_{44}^4 \Gamma_{4\mu}^\mu - \Gamma_{4\mu}^\lambda \Gamma_{4\lambda}^\mu - \Gamma_{4\mu}^4 \Gamma_{44}^\mu. \quad (8)$$

Evaluating the corresponding Christoffel symbols in Eq.(8) leads to

$$R_{44} = -\epsilon \Phi \square \Phi. \quad (9)$$

We now construct the space-time components of the Einstein tensor

$$G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R_{(5)}.$$

In so doing, we first obtain the  $5D$  Ricci scalar  $R_{(5)}$  as

$$\begin{aligned} R_{(5)} &= g^{AB}R_{AB} = \hat{g}^{\alpha\beta}\hat{R}_{\alpha\beta} + g^{44}R_{44} = g^{\alpha\beta}\left(R_{\alpha\beta} - \frac{\nabla_\alpha\nabla_\beta\Phi}{\Phi}\right) + \frac{\epsilon}{\Phi^2}(-\epsilon\Phi\Box\Phi) \\ &= R - \frac{2}{\Phi}\Box\Phi, \end{aligned} \tag{10}$$

where the  $\alpha 4$  terms vanish and  $R$  is the  $4D$  Ricci scalar. The space-time components of the Einstein tensor is written  $\hat{G}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2}\hat{g}_{\alpha\beta}R_{(5)}$ . Substituting  $\hat{R}_{\alpha\beta}$  and  $R_{(5)}$  into the space-time components of the Einstein tensor gives

$$\hat{G}_{\alpha\beta} = G_{\alpha\beta} + \frac{1}{\Phi}(g_{\alpha\beta}\Box\Phi - \nabla_\alpha\nabla_\beta\Phi). \tag{11}$$

In the same way, the 4-4 component is written  $G_{44} = R_{44} - \frac{1}{2}g_{44}R_{(5)}$ , and substituting  $R_{44}$ ,  $R_{(5)}$  into this component of the Einstein tensor gives

$$G_{44} = -\frac{1}{2}\epsilon R\Phi^2, \tag{12}$$

We now consider the  $5D$  energy-momentum tensor. The form of energy-momentum tensor is dictated by Einstein's equations and by the symmetries of the metric (2). Therefore, we may assume a perfect fluid with nonvanishing elements

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta}, \tag{13}$$

$$T_{44} = -\bar{p}g_{44} = -\epsilon\bar{p}\Phi^2, \tag{14}$$

where  $\rho$  and  $p$  are the conventional density and pressure of perfect fluid in the  $4D$  standard cosmology and  $\bar{p}$  acts as a pressure living along the higher dimensional sector. Hence, the

field equations (3) are to be viewed as *constraints* on the simultaneous geometric and physical choices of  $G_{AB}$  and  $T_{AB}$  components, respectively.

Substituting the energy-momentum components (13), (14) in front of the  $4D$  and extra dimensional part of Einstein tensors (11) and (12), respectively, we obtain the field equations<sup>1</sup>

$$G_{\alpha\beta} = 8\pi G[(\rho + p)u_\alpha u_\beta - pg_{\alpha\beta}] + \frac{1}{\Phi} [\nabla_\alpha \nabla_\beta \Phi - \square \Phi g_{\alpha\beta}], \quad (15)$$

and

$$R = 16\pi G\bar{p}. \quad (16)$$

By evaluating the  $g^{\alpha\beta}$  trace of Eq.(15) and combining with Eq.(16) we obtain

$$\square \Phi = \frac{1}{3}(8\pi G(\rho - 3p) + 16\pi G\bar{p})\Phi. \quad (17)$$

This equation infers the following scalar field potential

$$V(\Phi) = -\frac{1}{6}(8\pi G(\rho - 3p) + 16\pi G\bar{p})\Phi^2, \quad (18)$$

whose minimum occurs at  $\Phi = 0$ , for which the equations (15) reduce to describe a usual  $4D$  FRW universe filled with ordinary matter  $\rho$  and  $p$ . In other words, our conventional  $4D$  universe corresponds to the vacuum state of the scalar field  $\Phi$ . From Eq.(17), one may infer the following replacements for a nonvanishing  $\Phi$

$$\frac{1}{\Phi}\square\Phi = \frac{1}{3}(8\pi G(\rho - 3p) + 16\pi G\bar{p}), \quad (19)$$

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<sup>1</sup>The  $\alpha 4$  components of Einstein equation (3) result in

$$R_{\alpha 4} = 0,$$

which is an identity with no useful information.

$$\frac{1}{\Phi}\nabla_\alpha\nabla_\beta\Phi = \frac{1}{3}(8\pi G(\rho - 3p) + 16\pi G\bar{p})u_\alpha u_\beta. \quad (20)$$

Putting the above replacements into Eq.(15) leads to

$$G_{\alpha\beta} = 8\pi G[(\rho + \tilde{p})u_\alpha u_\beta - \tilde{p}g_{\alpha\beta}], \quad (21)$$

where

$$\tilde{p} = \frac{1}{3}(\rho + 2\bar{p}). \quad (22)$$

This energy-momentum tensor effectively describes a perfect fluid with density  $\rho$  and pressure  $\tilde{p}$ . It is very interesting that the contributions of the non-vacuum states of the scalar field at higher dimension cancels out exactly the physics of pressure  $p$  in four dimensions. The field equations lead to two independent equations

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi G\rho, \quad (23)$$

$$\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} = -8\pi G\tilde{p}. \quad (24)$$

Differentiating (23) and combining with (24) we obtain the conservation equation

$$\frac{d}{dt}(\rho a^3) + \tilde{p}\frac{d}{dt}(a^3) = 0. \quad (25)$$

The equations (23) and (25) can be used to derive the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\tilde{p}) = -\frac{8\pi G}{3}(\rho + \bar{p}). \quad (26)$$

If we choose the open universe ( $k = -1$ ) in Robertson-Walker metric (2) so that  $R = -a^{-2}$  and  $\bar{p} = -\frac{1}{16\pi G}a^{-2}$ , and insert a power law behavior  $\rho = Aa^\alpha$  into the conservation equation (25), then we obtain<sup>2</sup>

$$\rho = \frac{1}{16\pi G}a^{-2} > 0. \quad (27)$$

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<sup>2</sup>A close universe  $k = 1$ ,  $R > 0$  will result in  $\rho = -\frac{1}{16\pi G}a^{-2} < 0$  which is not physically viable.



By substituting  $\rho$  and  $\bar{p}$  into the acceleration equation (26) we find

$$\ddot{a} = 0. \tag{28}$$

Therefore, we conclude that the contributions of non-vacuum states of the scalar field, living along the higher dimension, can lead to zero acceleration of the  $4D$  universe, no matter which equation of state  $p = p(\rho)$  is used.

## Conclusion

In this paper, we have studied a  $(4 + 1)$ -dimensional metric subject to a  $(4 + 1)$  dimensional energy-momentum tensor in the framework of noncompact Kaluza-Klein theory. The  $4D$  part of the metric is taken to be Robertson-Walker one subject to the conventional perfect fluid with density  $\rho$  and pressure  $p$ , and the extra-dimensional part endowed by a scalar field is subject to the dark pressure  $\bar{p}$ . By writing down the reduced  $4D$  and extra-dimensional components of  $5D$  Einstein equations we found that our  $4D$  universe corresponds to the vacuum state of the scalar field. It turned out that the contributions of the non-vacuum states of the scalar field to the  $4D$  cosmology cancels out exactly the physics of pressure  $p$  in four dimensions and leads to zero acceleration of the  $4D$  universe for any equation of state. In other words, if the non-vacuum states of the scalar field at higher dimension would contribute to the  $4D$  cosmology then we would not see the current acceleration or even expect early inflation of the universe. It is then possible to think about other universes, living in excited states of the scalar field, in which neither inflation nor acceleration ever happens.

This model, although introduced in the framework of noncompact Kaluza-Klein theory,

is not of Space-time-matter type as Bellini *et al* have already worked out. So, a comparison between the approach and results of this model and those of Bellini *et al* is more constructive: In the Bellini *et al* approach, the Ricci flat assumption  $R_{AB} = 0$  is made where matter as a whole is induced by the *dynamics* of extra dimension. They developed a 5D mechanism to explain the ( neutral scalar field governed ) evolution of the universe from inflationary expansion towards a decelerated expansion followed by the present day observed accelerated expansion. In this model, however, we assumed a full 5D Einstein equation coupled to a higher dimensional energy momentum tensor whose components are all independent of 5th dimension and its one extra dimensional component is a scalar field. Reduction to four dimensions led us to 4D Einstein equation coupled to 4D energy momentum tensor ( perfect fluid ) accompanied by some terms of scalar field contribution induced from extra dimension. Both models from different approaches try to address the early inflation and current acceleration of the universe. Bellini *et al* explain both early inflation and current acceleration of the universe by a single scalar field. In the present model, however, we show that the contributions of non-vacuum states of a scalar field can destroy early inflation and current acceleration of the universe. This result is independent of the signature  $\epsilon$  by which the higher dimension takes part in the 5D metric.

Finally, we comment on the conceptual issue which is usually considered in higher dimensional theories: *why we perceive the 4 dimensions of space-time and apparently do not see the fifth dimension?* In old Kaluza-Klein theory this question is answered by resorting to a cyclic condition imposed on the 5th coordinate. Brane world cosmology also provides a mechanism by which matter and all but gravitational interactions stick to the branes. In modern Kaluza-

Klein theory (STM), however, the matter itself and the induced fifth force manifest as the direct results of the existence of the 5th dimension. Similarly, in the present model we find that the extra dimension may manifest through a dark pressure. However, as we discussed above the existence of this dark pressure, through non-vacuum states of the scalar field, will contradict the observed acceleration and even early inflation of the universe, So, it turns out that there is no such influence of dark pressure in our universe and the reason why we do not see the higher dimension in this model is that we are living in the vacuum state of the scalar field.

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## References

- [1] T. Appelquist, A. Chodos and P. G. O. Freund, *Modern Kaluza-Klein Theories*, Frontiers in Physics Series, (Volume **65**), 1986, (Ed. Addison-Wesely).
- [2] P. S. Wesson, Gen. Relativ. Gravit.**16**, 193 (1984); *Space-Time-Matter: Modern Kaluza-Klein Theory*, (World Scientific. Singapore 1999).
- [3] L. Randall, R. Sundrum, Phys. Rev. Lett.**83**, 4690 (1999); Phys. Rev. Lett. **83**, 3370 (1999).
- [4] P. Horava, E. Witten, Nucl. Phys. B**475**, 94 (1996); Nucl. Phys. B**460**, 506 (1996).
- [5] S. Perlmutter *et al.*, Bull. Am. Phys. Soc. **29**, 1351 (1997); Ap. J. **507**, 46 (1998); A. G. Riess *et al.*, Astron. J. **116** (1998); P. M. Garnavich *et al.*, Ap. J. Lett. **493**, 53 (1998); Science **279**, 1298 (1998); Ap. J. **509**, 74 (1998); B. Schmidt *et al.*, *The High-Z Supernova Search: Measuring Cosmic Deceleration and Global Curvature of the Universe using Type IA Supernova*, astro-ph/9805200.
- [6] L. Krauss and M. S. Turner, Gen. Rel. Grav. **27**, 1137 (1995); J. P. Ostriker and P. J. Steinhardt, Nature **377**, 600 (1995); A. R. Liddle, D. H. Lyth, P. T. Viana and M. White, Mon. Not. Roy. Astron. Soc. **282**, 281 (1996).
- [7] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
- [8] A. A. Starobinsky, JETP Lett.**68**, 757 (1998) ; T. D. Saini, S. Raychaudhury, V. Sahni, A. A. Starobinsky, Phys. Rev. Lett.**85**, 1162 (2000).

- [9] A. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. **B511**, 265 (2001); N. Bilic, G. B. Tupper and R. Viollier, Phys. Lett. **B535**, 17 (2002); J. S. Fabris, S.V. Goncalves and P.E. de Souza, Gen. Rel. Grav. **34**, 53 (2002); R. Jackiw, *A particle field theorists lectures on supersymmetric, nonabelian fluid mechanics and D-branes*, physics/0010042.
- [10] D. Bazeia, R. Jackiw, Ann. Phys. **270**, 246 (1998); D. Bazeia, Phys. Rev. D **59**, 085007 (1999); R. Jackiw, A. P. Polychronakos, Commun. Math. Phys. **207**, 107 (1999); N. Ogawa, Phys. Rev. D **62**, 085023 (2000).
- [11] J. E. Madriz Aguilar, M. Bellini, Phys. Lett. **B596**, 116 (2004); J. E. Madriz Aguilar, M. Bellini, Eur. Phys. J. C **38**, 367 (2004); M. Bellini, Nucl. Phys. B **660**, 389 (2003); J. Ponce de Leon, Int. J. Mod. Phys. D **15**, 1237 (2006).
- [12] P. S. Wesson, J. Ponce de Leon, J. Math. Phys. **33**, 3883, (1992); J. Ponce de Leon, P. S. Wesson, J. Math. Phys. **34**, 4080, (1993).